6 Line Segments and Rays

Definition (line segment $\overline{AB}$) If A and B are distinct points in a metric geometry $\{\mathcal{S},\mathcal{L},d\}$ then the line segment from A to B is the set $\overline{AB}=\{M\in\mathcal{S}\mid A-M-B \text{ or } M=A \text{ or } M=B\}$.

1. Let $A(-1/2,\sqrt{3}/2)$ and $B(\sqrt{19}/10,1/10)$ denote given points of line ${}_0L_1$. Give a graphical sketch for line segment $\overline{AB}$.

2. Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ denote three points which belong to the type II line ${}_cL_r$ in the Poincaré Plane. If $x_1<x_3<x_2$ show that then $C\in\overline{AB}$.

3. Let $A(x_1,y_1)$ and $B(x_2,y_2)$ lie on the type II line ${}_cL_r$ in the Poincaré Plane. If $x_1<x_2$ show that $\overline{AB}=\{C=(x,y)\in {}_cL_r\mid x_1\le x\le x_2\}$.

Definition Let $\mathcal{A}$ be a subset of a metric geometry. A point $B\in\mathcal{A}$ is a passing point of $\mathcal{A}$ if there exists points $X,Y\in\mathcal{A}$ with $X-B-Y$. Otherwise B is an extreme point of $\mathcal{A}$.

4. Let $A(x_1,y_1)$ and $B(x_2,y_2)$ denote two points in metric geometry, and let $C\in\overline{AB}$. If $C\neq A$ and $C\neq B$ explain is point C passing point or extreme point of $\overline{AB}$.

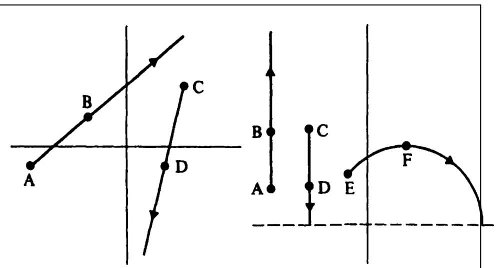
Theorem If A and B are two points in a metric geometry then the only extreme points of the segment $\overline{AB}$ are A and B themselves. In particular, if $\overline{AB}=\overline{CD}$ then $\{A,B\}=\{C,D\}$.

5. Prove the above theorem. [use proof by contradiction to show that A is not a passing point of $\overline{AB}$...]

Definition (end points, length of the segment $\overline{AB}$) The end points (or vertices) of the segment $\overline{AB}$ are A and B. The length of the segment $\overline{AB}$ is $AB=d(A,B)$.

Definition (ray $pp[A,B)=\overrightarrow{AB}$)
If A and B are distinct points in a metric geometry $\{\mathcal{S},\mathcal{L},d\}$ then the ray from A toward B is the set

$$pp[A,B)=\overrightarrow{AB}=\overline{AB}\cup\{C\in\mathcal{S}\mid A-B-C\}.$$



Definition (vertex of the ray) The vertex (or initial point) of the ray $pp[A,B)=\overrightarrow{AB}$ is the point A.

Theorem If A and B are distinct points in a metric geometry then there is a ruler $f:\overleftrightarrow{AB}\to\mathbb{R}$ such that $pp[A,B)=\overrightarrow{AB}=\{X\in\overleftrightarrow{AB}\mid f(X)\ge 0\}$

6. Prove the above theorem. [$\{X\in\overleftrightarrow{AB}\mid f(X)\ge 0\}\subseteq\overrightarrow{AB}$; $\overrightarrow{AB}\subseteq\{X\in\overleftrightarrow{AB}\mid f(X)\ge 0\}$]

Definition ($\overline{AB}\cong\overline{CD}$) Two line segments $\overline{AB}$ and $\overline{CD}$ in a metric geometry are congruent (written $\overline{AB}\cong\overline{CD}$) if their lengths are equal; that is $\overline{AB}\cong\overline{CD}$ if $AB=CD$.

Theorem (Segment Construction). If $\overrightarrow{AB}$ is a ray and $\overline{PQ}$ is a line segment in a metric geometry, then there is a unique point $C\in\overrightarrow{AB}$ with $\overline{PQ}\cong\overline{AC}$.

7. Prove the above theorem. [let f be coordinate system for $\overrightarrow{AB}$ with A as origin and B positive...]

8. In the Poincaré Plane let $A(0,2)$, $B(0,1)$, $P(0,4)$, $Q(7,3)$. Find $C\in\overrightarrow{AB}$ so that $\overline{AC}\cong\overline{PQ}$.

9. Let A and B be distinct points in a metric geometry. Then $M\in\overleftrightarrow{AB}$ is a midpoint of the line segment $\overline{AB}$ if and only if $AM=MB$. (Remember that here AM means $d(A,M)$.) (a) If M is a midpoint of $\overline{AB}$, prove that $A-M-B$. (b) Show that $\overline{AB}$ has a midpoint M, and that M is unique. (c) Let $A(0,9)$ and $B(0,1)$. Find the midpoint of $\overline{AB}$ where A and B are points of (i) the Euclidean plane; (ii) the Hyperbolic plane.

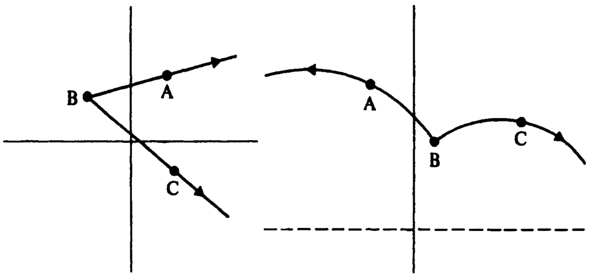
10. determine are the statements true or false: (a) $\overline{AB} = \overline{CD}$ only if $A = C$ or $A = D$. (b) If $AB = CD$ then $A = C$ or $A = D$. (c) If $\overline{AB} \cong \overline{CD}$, then $\{A, B\} = \{C, D\}$. (d) If $\overline{AB} \cong \overline{CD}$, then $\overline{AB} = \overline{CD}$. (e) A point on $\overleftrightarrow{AB}$ is uniquely determined by its distances from A and B.

11. In a metric geometry prove that (i) if $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$; (ii) if $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$.

12. In a metric geometry $(\mathcal{S}, \mathcal{L}, d)$, prove that if $A - B - C$, $P - Q - R$, $\overline{AB} \cong \overline{PQ}$, $\overline{AC} \cong \overline{PR}$, then $\overline{BC} \cong \overline{QR}$.

7 Angles and Triangles

It is important to note that an angle is a set, not a number like 45°. We will view numbers as properties of angles when we define angle measure in section: "The Measure of an Angle".



Definition (angle $\measuredangle ABC$)
If A, B and C are noncollinear points in a metric geometry then the angle $\measuredangle ABC$ is the set

$$\measuredangle ABC = \overrightarrow{BA} \cup \overrightarrow{BC} = pp[B, A) \cup pp[B, C).$$

Lemma In a metric geometry, B is the only extreme point of $\measuredangle ABC$.

1. Prove the above lemma. [$Z \in \measuredangle ABC$, $Z \neq B \Rightarrow Z$ is a passing point of $\measuredangle ABC$...]

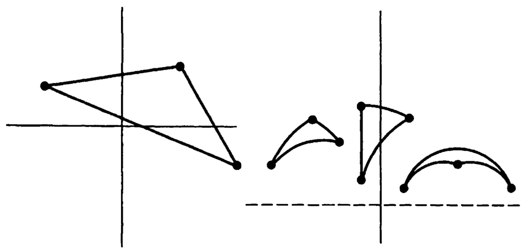
Theorem ($\measuredangle ABC = \measuredangle DEF \Rightarrow B = E$) In a metric geometry, if $\measuredangle ABC = \measuredangle DEF$ then $B = E$.

2. Prove the above theorem. [$\{B\} = ... = \{E\}$]

Definition (vertex of the angle $\measuredangle ABC$)
The vertex of the angle $\measuredangle ABC$ in a metric geometry is the point B.

Definition (triangle $\triangle ABC$)
If $\{A, B, C\}$ are noncollinear points in a metric geometry then the triangle $\triangle ABC$ is the set

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}.$$



Lemma In a metric geometry, if A, B, and C are not collinear then A is an extreme point of $\triangle ABC$.

3. Prove the above lemma. [proof is by contradiction, suppose that $D - A - E$ with $D, E \in \triangle ABC$...]

Theorem In a metric geometry, if $\triangle ABC = \triangle DEF$ then $\{A, B, C\} = \{D, E, F\}$.

4. Prove the above theorem. [If $X \in \triangle ABC$ and $X \notin \{A, B, C\}$ then X is in one the segments...]

Definition (vertices, sides) In a metric geometry the vertices of $\triangle ABC$ are the points A, B, C. The sides (or edges) of $\triangle ABC$ are $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

5. Prove that $\measuredangle ABC = \measuredangle CBA$ in a metric geometry.

In next two problems do not use last Lemma and last Theorem above.

6. Prove that if $\triangle ABC = \triangle DEF$ in a metric geometry then $\overleftrightarrow{AB}$ contains exactly two of the points D, E and F.

7. In a metric geometry, prove that if A, B and C are not collinear then $\overline{AB} = \overleftrightarrow{AB} \cap \triangle ABC$.

Duži i poluprave

Definicija (duž)

Ako su A i B dvije različite tačke u metričkoj geometriji $\{\mathcal{S}, \mathcal{L}, d\}$ tada je duž od A do B skup

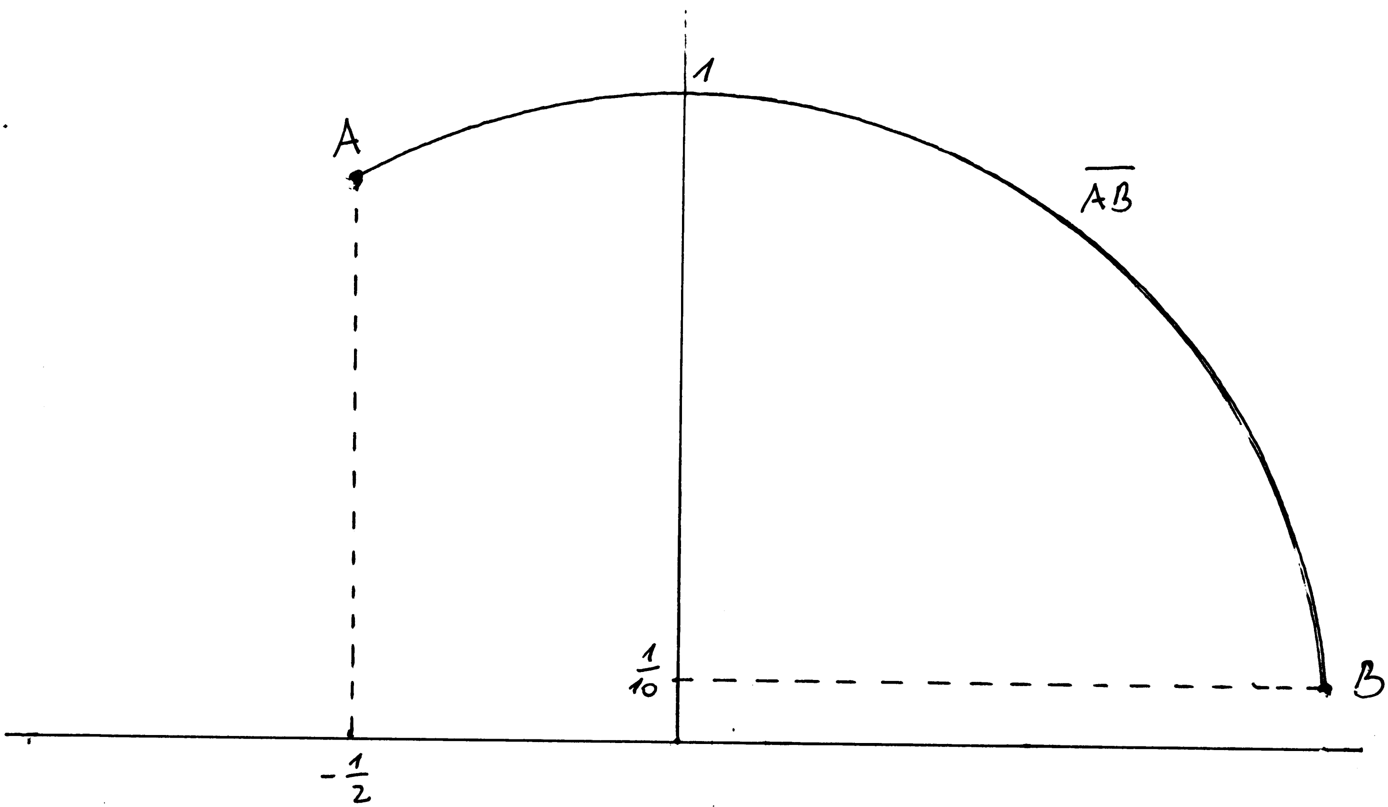
$$\overline{AB} = \{C \in \mathcal{S} \mid A-C-B \text{ ili } C=A \text{ ili } C=B\}$$

(#) Date su tačke $A\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ i $B\left(\frac{\sqrt{19}}{10}, \frac{1}{10}\right)$ prave ${}_0L_1$. Skicirati duž $\overline{AB}$.

Rj:

$${}_0L_1 : \quad (x-0)^2 + y^2 = 1^2$$

$$x^2 + y^2 = 1$$



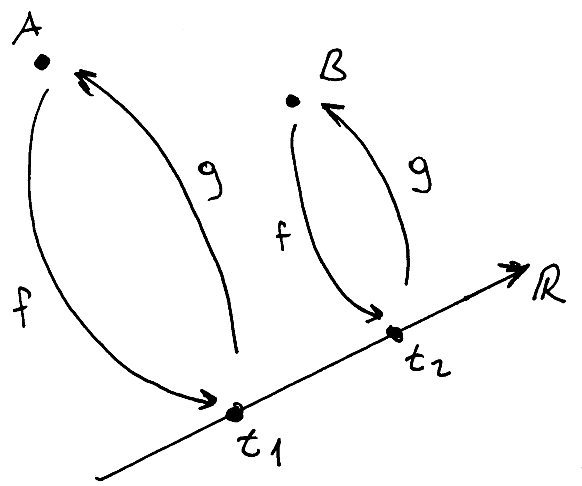
Neka su $A(x_1,y_1)$, $B(x_2,y_2)$ i $C(x_3,y_3)$ tri tačke koje pripadaju pravoj tipa II ${}_cL_r$ u Poincaré-ovoj ravni. Ako je $x_1<x_3<x_2$ pokazati da tada $C\in\overline{AB}$.

Rj.

Standardna mjera za pravu ${}_cL_r$ je
$$f: {}_cL_r \to \mathbb{R}$$
$$(x,y)\to \ln\left(\frac{x-c+r}{y}\right).$$

Inverz f-je f je
$$g:\mathbb{R}\to {}_cL_r$$
$$t \to (c+r\tanh(t),\ r\,\mathrm{sech}(t))$$

Time, grafički imamo sljedeće



gdje je $f(A)=t_1$, $f(B)=t_2$ i neka je $f(C)=t_3$

Primjetimo da je
$$g(t_1)=(x_1,y_1)$$
$$g(t_2)=(x_2,y_2)$$
$$g(t_3)=(x_3,y_3)$$

kako je
$$g(t_1)=(c+r\tanh(t_1),\ r\,\mathrm{sech}(t))$$
$$g(t_2)=(c+r\tanh(t_2),\ r\,\mathrm{sech}(t))$$
$$g(t_3)=(c+r\tanh(t_3),\ r\,\mathrm{sech}(t))$$

to je
$$\left.\begin{aligned} x_1&=c+r\tanh(t_1)\\ x_2&=c+r\tanh(t_2)\\ x_3&=c+r\tanh(t_3)\end{aligned}\right\}\ldots(1).$$

Sad
$$x_1<x_3<x_2 \overset{(1)}{\Longrightarrow} r\tanh(t_1)<r\tanh(t_3)<r\tanh(t_2) \Rightarrow$$

$$t_1 < t_3 < t_2 \implies \underbrace{t_1}_{=f(A)} * \underbrace{t_3}_{=f(C)} * \underbrace{t_2}_{=f(B)} \overset{\text{prem Teor. 17 pretk. lekcji}}{\iff}$$

$$\overset{?}{\iff} \quad A - C - B \implies C \in \overline{AB}$$

q.e.d.

(#) Neka su $A(x_1,y_1)$ i $B(x_2,y_2)$ tačke koje pripadaju pravoj tipa II ${}_cL_r$ u Poincaré-ovoj ravni. Ako je $x_1<x_2$ pokazati da

$$\overline{AB}=\{C(x,y)\in {}_cL_r \mid x_1\le x\le x_2\}$$

Rj.

$${}_cL_r : (x-c)^2+y^2=r^2$$

Ako posmatramo tačku C čija je prva koordinata x_1 tada

$$y^2=r^2-(x_1-c)^2 \Rightarrow y=y_1 \Rightarrow C(x_1,y_1) \Rightarrow C=A$$

$$x=x_2 \Rightarrow C(x_2,y_2) \Rightarrow C=B.$$

Ostalo je da pokažemo da $A-C-B$ ako i samo ako $x_1<x<x_2$ (gdje je $C(x,y)$).

Standardna mjera $f: {}_cL_r\to\mathbb{R}$ za pravu tipa II u Poincaré-ovoj ravni je

$$f(x,y)=\ln\left(\frac{x-c+r}{y}\right).$$

U jednom od prethodnih zadataka smo odredili inverz bijekcije f: inverz je

$$g:\mathbb{R}\to {}_cL_r$$
$$t\to(c+r\,th(t),\ r\,sech(t))$$

Iskoristimo ove dvije f-je u rješavanju zadatka.

Pretpostavimo da je $A-C-B$. Prema teoremu iz prethodne lekcije

$$A-C-B \Leftrightarrow f(A)*f(C)*f(B)$$

Označimo vrijednosti $f(A)$, $f(C)$ i $f(B)$ redom sa t_1, t_3 i t_2

$$f(A)*f(C)*f(B) \Leftrightarrow t_1*t_3*t_2$$

F-ja $th(t)$ je strogo rastuća, pa nije važno da li je $t_1 < t_3 < t_2$ ili $t_2 < t_3 < t_1$, imamo da

$$t_1 * t_3 * t_2 \overset{?}{\iff} (c + r\, th(t_1)) * (c + r\, th(t_3)) * (c + r\, th(t_2))$$

Kako je

$$g(t_1) = (c + r\, th(t_1), r\, sech(t_1)) = A, \qquad A(x_1, y_1)$$
$$g(t_2) = (c + r\, th(t_2), r\, sech(t_2)) = B, \qquad B(x_2, y_2)$$

to je

vidi prethodni zadatak

$$(c + r\, th(t_1)) * (c + r\, th(t_3)) * (c + r\, th(t_2)) \overset{?}{\iff} x_1 * x * x_2$$

Kako je prema pretpostavci $x_1 < x_2$ to je $x_1 < x < x_2$.

Time smo pokazali da

$$C \in \overline{AB} \iff x_1 \leq x \leq x_2$$
(gdje je $C(x, y)$)

<u>Definicija</u> (prolazna tačka, ekstremna tačka)

Neka je $\mathcal{A}$ podskup metričke geometrije. Tačka $B \in \mathcal{A}$ je prolazna tačka skupa $\mathcal{A}$ ako postoje tačke X i $Y \in \mathcal{A}$ sa osobinom $X-B-Y$. U suprotnom B je ekstremna tačka skupa $\mathcal{A}$.

Neka su A i B dvije tačke metričke geometrije i neka je $C \in \overline{AB}$. Ako je $C \neq A$ i $C \neq B$ objasniti da li je tačka C prolazna tačka ili ekstremna tačka duži $\overline{AB}$.

Rj.

$$\overline{AB} \stackrel{def}{=} \{ D \in \mathcal{S} \mid A-D-B \text{ ili } D=A \text{ ili } D=B \}$$

Ako je $C \in \overline{AB}$ i $C \neq A$ i $C \neq B$ tada $A-C-B$.

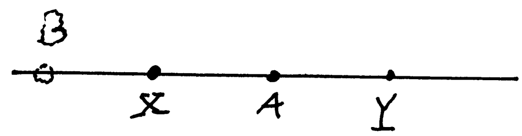
Drugim riječima postoje tačke $X, Y \in \mathcal{S}$ t.d. $X-C-Y$ $(X=A,\ Y=B)$.

Tačka C je prolazna tačka duži $\overline{AB}$.

Teorema

Ako su A i B dvije tačke u metričnoj geometriji tada su jedine ekstremne tačke duži $\overline{AB}$ same tačke A i B. Preciznije, ako je $\overline{AB} = \overline{CD}$ tada je $\{A, B\} = \{C, D\}$.

Skica dokaza:



koristimo dokaz kontradikcijom

A između X, Y duži $\overline{AB}$, $X-A-Y$

$$\left.\begin{array}{lll} 1^\circ & B-X-A-Y \Rightarrow & B-A-Y \\ 2^\circ & B=X \Rightarrow & B-A-Y \\ 3^\circ & X-B-A-Y \Rightarrow & B-A-Y \end{array}\right\} \Rightarrow Y \notin \overline{AB}$$

\# kontradikcija ($X, Y \in \overline{AB}$)

$$\left.\begin{array}{lll} 4^\circ & X-A-B-Y \Rightarrow & X-A-B \\ 5^\circ & B=Y \Rightarrow & X-A-B \\ 6^\circ & X-A-Y-B \Rightarrow & X-A-B \end{array}\right\} \Rightarrow X \notin \overline{AB}$$

\# kontradikcija ($X, Y \in \overline{AB}$)

Slično za tačku B. $\Rightarrow$ A i B su ekstremne tačke

Pokažimo da ni jedna druga tačka nije prolazna tačka.

$Z \in \overline{AB}$, $Z \neq A$, $Z \neq B$ $\Rightarrow$ $A-Z-B$ $\Rightarrow$ Z između A i B

$\Rightarrow$ Z prolazna tačka

A, B su dvije jedine ekstremne tačke duži $\overline{AB}$

$$\begin{aligned} \overline{AB} = \overline{CD} \Rightarrow \{A, B\} &= \{Z \in \overline{AB} \mid Z \text{ je ekstremna tačka duži } \overline{AB}\} \\ &= \{Z \in \overline{CD} \mid Z \text{ je ekstremna tačka duži } \overline{CD}\} \\ &= \{C, D\}. \end{aligned}$$

Definicija (krajnje tačke)

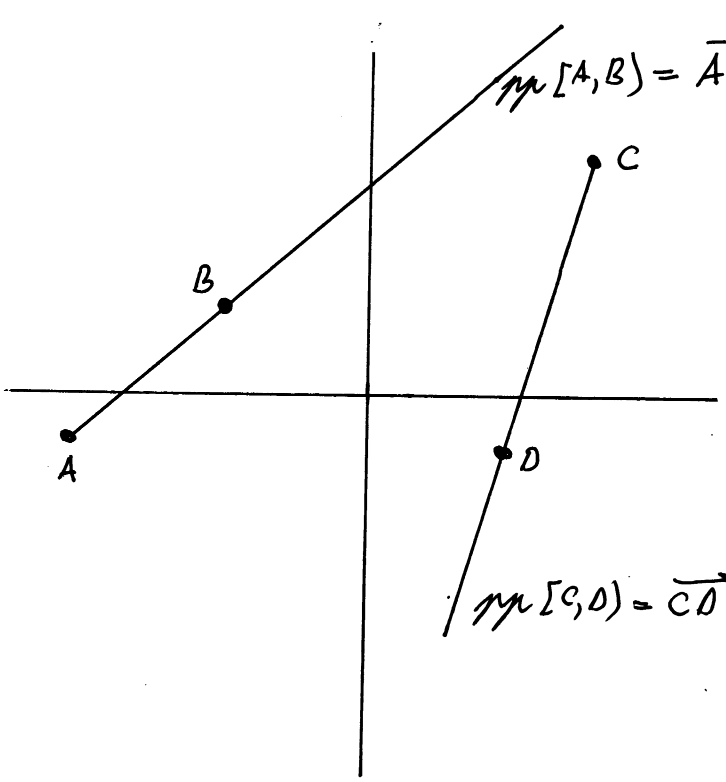
Krajnje tačke (ili krajnji vrhovi) duži $\overline{AB}$ su tačke A i B.
Dužinu duži $\overline{AB}$ definišemo sa $AB = d(A,B)$.

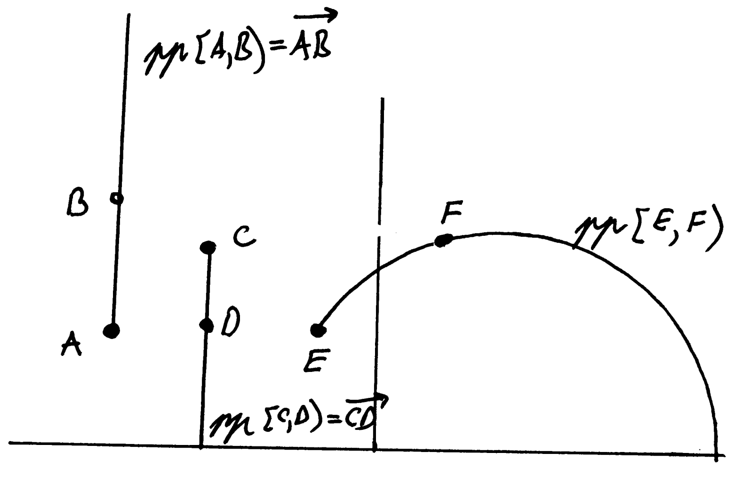
Definicija (poluprava)

Ako su A i B različite tačke metričke geometrije $\{\mathcal{S}, \mathcal{L}, d\}$ tada poluprava sa početnom tačkom A koja sadrži tačku B je skup

$$pp[A,B) = \overrightarrow{AB} = \overline{AB} \cup \{C \in \mathcal{S} \mid A-B-C\}.$$

Primjetimo da je poluprava $pp[A,B) = \overrightarrow{AB}$ podskup prave $p(A,B) = \overleftrightarrow{AB}$. Poluprave u Euklidovoj i Poincaré-ovoj ravni su prikazane na sljedećim slikama





Definicija (vrh ili početna tačka)

Vrh (ili početna tačka) poluprave $\overrightarrow{AB} = pp[A, B)$ je tačka A.

Teorema

Ako su A i B dvije različite tačke metrične geometrije tada postoji mjera $f : \overleftrightarrow{AB} \to \mathbb{R}$ takva da

$$\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} \mid f(X) \geq 0\}$$

Skica dokaza:

f spec. koor. sist. $f(A)=0$, $f(B)>0$

$\{X \in \overleftrightarrow{AB} \mid f(X) \geq 0\} \subseteq \overrightarrow{AB}$

$X \in \overleftrightarrow{AB}$, $f(X) \geq 0$

$x = f(X)$, $f(B) = y$

$x = 0 \Rightarrow X = A \Rightarrow X \in \overrightarrow{AB}$

$x = y \Rightarrow X = B \Rightarrow X \in \overrightarrow{AB}$

$0 < x < y \Rightarrow A-X-B \Rightarrow X \in \overline{AB} \Rightarrow X \in \overrightarrow{AB}$

$0 < y < x \Rightarrow A-B-X \Rightarrow X \in \overrightarrow{AB}$

$\overrightarrow{AB} \subseteq \{X \in \overleftrightarrow{AB} \mid f(X) \geq 0\}$

$D \in \overrightarrow{AB}$ $(D \in \overleftrightarrow{AB})$, $x = f(D) < 0$

$f(A)=0$, $f(B)=y>0$, $x<0<y$. $\Rightarrow$ $D-A-B$
kontradikcija
$(D \in \overrightarrow{AB})$

Time svi elementi od $\overrightarrow{AB}$ imaju nenegativne koordinate u odnosu na f.

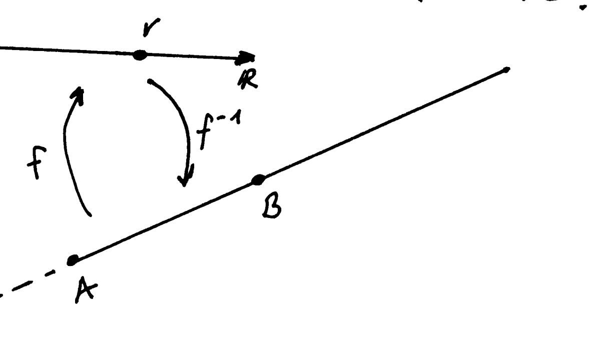
Definicija

Za dvije duži $\overline{AB}$ i $\overline{CD}$ u metričnoj geometriji kažemo da su kongruentne (što pišemo $\overline{AB} \cong \overline{CD}$) ako su njihove dužine jednake: tj.

$$\overline{AB} \cong \overline{CD} \quad \text{ako} \quad AB = CD$$

Teorema (konstrukcija duži)

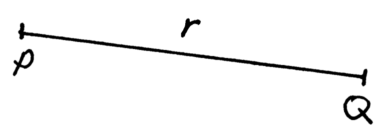
Ako su $\overrightarrow{AB}$ polupravа i $\overline{PQ}$ duž u metričnoj geometriji, tada postoji jedinstvena tačka $C \in \overrightarrow{AB}$ taka da $\overline{PQ} \cong \overline{AC}$.



Skica dokaza:

f spec. koor. sist. $\overleftrightarrow{AB}$, $f(A)=0$, $f(B)>0$

$\Rightarrow \overrightarrow{AB} = \{X \in \overleftrightarrow{AB} \mid f(X) \geq 0\}$



$r = PQ$, $C = f^{-1}(r)$

$r = PQ > 0 \Rightarrow C \in \overrightarrow{AB}$

$AC = |f(A) - f(C)| = |0 - r| = r = PQ \Rightarrow \overline{AC} \cong \overline{PQ}$

$\Rightarrow$ postoji najmanje jedna tačka $C \in \overrightarrow{AB}$ t. d. $\overline{AC} \cong \overline{PQ}$

$C' \in \overrightarrow{AB}$, $\overline{AC'} \cong \overline{PQ}$

$C' \in \overrightarrow{AB} \Rightarrow f(C') > 0$; $f(C') = f(C') - f(A) = |f(C') - f(A)|$

$= AC' = PQ = f(C)$

f injekcija $\Rightarrow C' = C$

Prema tome postoji tačno jedna tačka $C \in \overrightarrow{AB}$ taka da $\overline{AC} \cong \overline{PQ}$.

U Poincaréovoj ravni date su tačke $A(0,2)$, $B(0,1)$, $P(0,4)$ i $Q(7,3)$. Odrediti tačku $C \in \overrightarrow{AB}$ takvu da $\overline{AC} \cong \overline{PQ}$.

Rj:

Prvo odredimo PQ.

$$P,Q \in {}_{3}L_{5} \Rightarrow PQ = d_H(P,Q) = \left| \ln \frac{-3+5}{4} - \ln \frac{4+5}{3} \right| = \left| \ln \frac{1}{6} \right| = \ln 6$$

Primjetimo da $A, B \in {}_{0}L$.

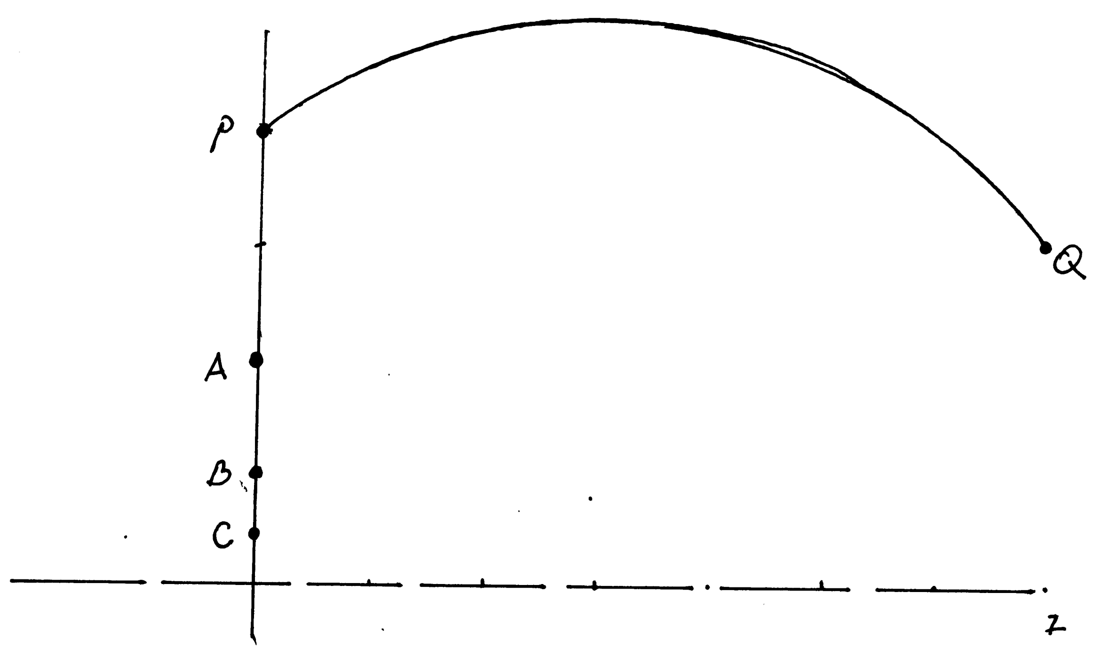
Kako je $C(0,Y)$ na pravoj tipa I prave $\overleftrightarrow{AB} = \mu(A,B)$

$$d_H(A,C) = \left| \ln \frac{Y}{2} \right|$$

Da bi vrijedilo $\overline{AC} \cong \overline{PQ}$ treba nam $\ln \frac{Y}{2} = \pm \ln 6$. Time

$$\frac{Y}{2} = 6 \quad \text{ili} \quad \frac{Y}{2} = \frac{1}{6} \quad \Rightarrow \quad Y = 12 \quad \text{ili} \quad Y = \frac{1}{3}$$

Kako je $C \in \overrightarrow{AB}$ treba nam $C = (0, \frac{1}{3})$.



9. Let A and B be distinct points in a metric geometry. Then $M \in \overleftrightarrow{AB}$ is a midpoint of the line segment $\overline{AB}$ if and only if $AM = MB$. (Remember that here AM means $d(A,M)$.) (a) If M is a midpoint of $\overline{AB}$, prove that $A-M-B$. (b) Show that $\overline{AB}$ has a midpoint M, and that M is unique. (c) Let $A(0,9)$ and $B(0,1)$. Find the midpoint of $\overline{AB}$ where A and B are points of (i) the Euclidean plane; (ii) the Hyperbolic plane.

M is midpt of $\overline{AB}$ iff $AM = MB$ (and $M \in \overleftrightarrow{AB}$) (here AM means $d(A,M)$).

(a) Claim: $A-M-B$.

Since A, M, B are collinear, one of the following holds:

$A-M-B$ or $A-B-M$ or $M-A-B$.

Now $A-B-M$ means $AM = AB + BM$, but $AM = MB$ so $AB = 0$ or $A = B$, not so.

And $M-A-B$ means $MA + AB = MB$, but $MB = MA$, so $AB = 0$, or $A = B$, not so.

Hence $A-M-B$, as required.

(b) Claim: $\overline{AB}$ has a midpt M, and M is unique. (Let $l = \overleftrightarrow{AB}$.)

Let $f: l \to \mathbb{R}$ be ruler for l with $f(A) = 0$, $f(B) = b > 0$.

Define $M \in \overleftrightarrow{AB}$ by $M = f^{-1}(b/2)$. Since f is onto $\mathbb{R}$, M exists, and since f is 1-1, M is uniquely determined.

And $d(A,M) = |f(A) - f(M)| = |0 - b/2| = b/2$,

while $d(M,B) = |f(M) - f(B)| = |b/2 - b| = b/2$.

Hence $AM = MB$, and M is the unique midpt. of $\overline{AB}$.

(c) $A = (0,9)$ and $B = (0,1)$. Want midpt of $\overline{AB}$ in

(i) $\mathcal{E}$, Euclidean plane; (ii) $\mathcal{H}$, Hyperbolic plane.

(i) Noting A, B are on L_0, and $\frac{9-1}{2} = 4$,

$M = (0,5)$ because

$d_E((0,9),(0,5)) = |9-5| = 4$ and $d_E((0,5),(0,1)) = |5-1| = 4$.

(ii) Line is ${}_0L$ in $\mathcal{H}$. Let M be $(0,m)$.

Then $f(M) = \ln(m)$, and we want

$|\ln m - \ln 1| = |\ln 9 - \ln m|$

or $\ln m = \ln \frac{9}{m}$, $m^2 = 9$, $m = 3$.

(Check: $\ln \frac{9}{3} = \ln \frac{3}{1}$)

So $(0,3) = M$ is the midpt. in $\mathcal{H}$.

SECTION A True/False *(10 marks)*

Tick ONE box for each question, according as the statement is true or false.
All the questions below are about a metric geometry.

Reminder: In a *metric* geometry, $\overleftrightarrow{AB}$ denotes the line containing points A and B; $\overline{AB}$ denotes the line segment from A to B, and AB denotes the distance $d(A, B)$. Two line segments are congruent if and only if their lengths are equal.

		TRUE	FALSE
A1	$\overline{AB} = \overline{CD}$ only if $A = C$ or $A = D$.	✓	
A2	If $AB = CD$ then $A = C$ or $A = D$.		✓
A3	If $\overline{AB} \simeq \overline{CD}$, then $\{A, B\} = \{C, D\}$.		✓
A4	If A—B—C and C—D—E, then A—C—D.		✓
A5	A point on $\overleftrightarrow{AB}$ is uniquely determined by its distances from A and B.	✓	
A6	If A, B are points, then $\overline{AB}$ is a convex set.	✓	
A7	If A, B are points, then $\{A, B\}$ is a convex set.		✓
A8	The intersection of two convex sets is a convex set.	✓	
A9	The union of two convex sets is a convex set.		✓
A10	$\overline{BC} = \overleftrightarrow{BC} \cap \triangle ABC$.	✓	

Working space

11. In a metric geometry prove that

(a) if $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$.

(b) if $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$;

SOLUTION:

(a) This came before Theorem 6.17, so if you don't use the fact that we could choose a ruler f for $\overleftrightarrow{AB}$ with $f(A) = 0$ and $f(B) > 0$, it's a bit long!

We have $\overrightarrow{AB} = \overline{AB} \cup \{P \in \mathcal{S} \mid A—B—P\}$, and $\overrightarrow{AB} = \overline{CD} \cup \{Q \in \mathcal{S} \mid C—D—Q\}$.

Since $\overrightarrow{AB} = \overrightarrow{CD}$, we know that $A \in \overrightarrow{CD}$. Suppose $A \neq C$. Then possibilities are:

(i) $C—A—D$; (ii) $A = D$; (iii) $C—D—A$.

Also $B \in \overrightarrow{CD}$, so (a) $C—B—D$ or (b) $C—D—B$ or (c) $B = D$ or (d) $B = C$.

Putting these together gives 12 cases in all:

(i)	$B = C$ and $C—A—D$	(ii)	$C—B—A—D$	(iii)	$C—A—B—D$
(iv)	$B = D$ and $C—A—D$	(v)	$C—A—D—B$	(vi)	$A = D$ and $C—B$
(vii)	$A = D$ and $C—A—B$	(viii)	$B = C$ and $C—D—A$	(ix)	$C—B—D—A$
(x)	$B = D$ and $C—D—A$	(xi)	$C—B—D—A$	(xii)	$C—D—A—B$.

On cases (i),(ii),(vi),(viii),(ix),(x),(xi), choose points E, F so that $E—B—A$ and $C—D—F$. Then we must have $E \in \overrightarrow{AB}$ and $F \in \overrightarrow{CD}$, but we may choose $E \notin \overrightarrow{CD}$ and $F \notin \overrightarrow{AB}$. Hence $\overrightarrow{AB} \neq \overrightarrow{CD}$, giving a contradiction.

In cases (iii),(iv),(v),(vii), we may choose E so that $C—E—A$; then $E \in \overrightarrow{CD}$ but $E \notin \overrightarrow{AB}$. Hence $\overrightarrow{AB} \neq \overrightarrow{CD}$, giving a contradiction.

In (xii), $A \notin \overrightarrow{CD}$, implying $A \notin \overrightarrow{AB}$, again a contradiction.

Hence $A = C$ as required.

(b) This is a bit long. Solution available on request!

12. In a metric geometry $(\mathcal{S}, \mathcal{L}, d)$, prove that if $A-B-C$, $P-Q-R$, $\overline{AB} \cong \overline{PQ}$, $\overline{AC} \cong \overline{PR}$, then $\overline{BC} \cong \overline{QR}$.

SOLUTION:
From A—B—C, we have A, B, C are collinear and (distances) $AB + BC = AC$. Likewise, P—Q—R means P, Q, R are collinear and $PQ + QR = PR$.

Now $\overline{AB} \simeq \overline{PQ}$ means $AB = PQ$, and $\overline{AC} \simeq \overline{PR}$ means $AC = PR$.

Hence $BC = AC - AB \ = \ PR - PQ \ = \ QR$, so $\overline{BC} \simeq \overline{QR}$, as required.

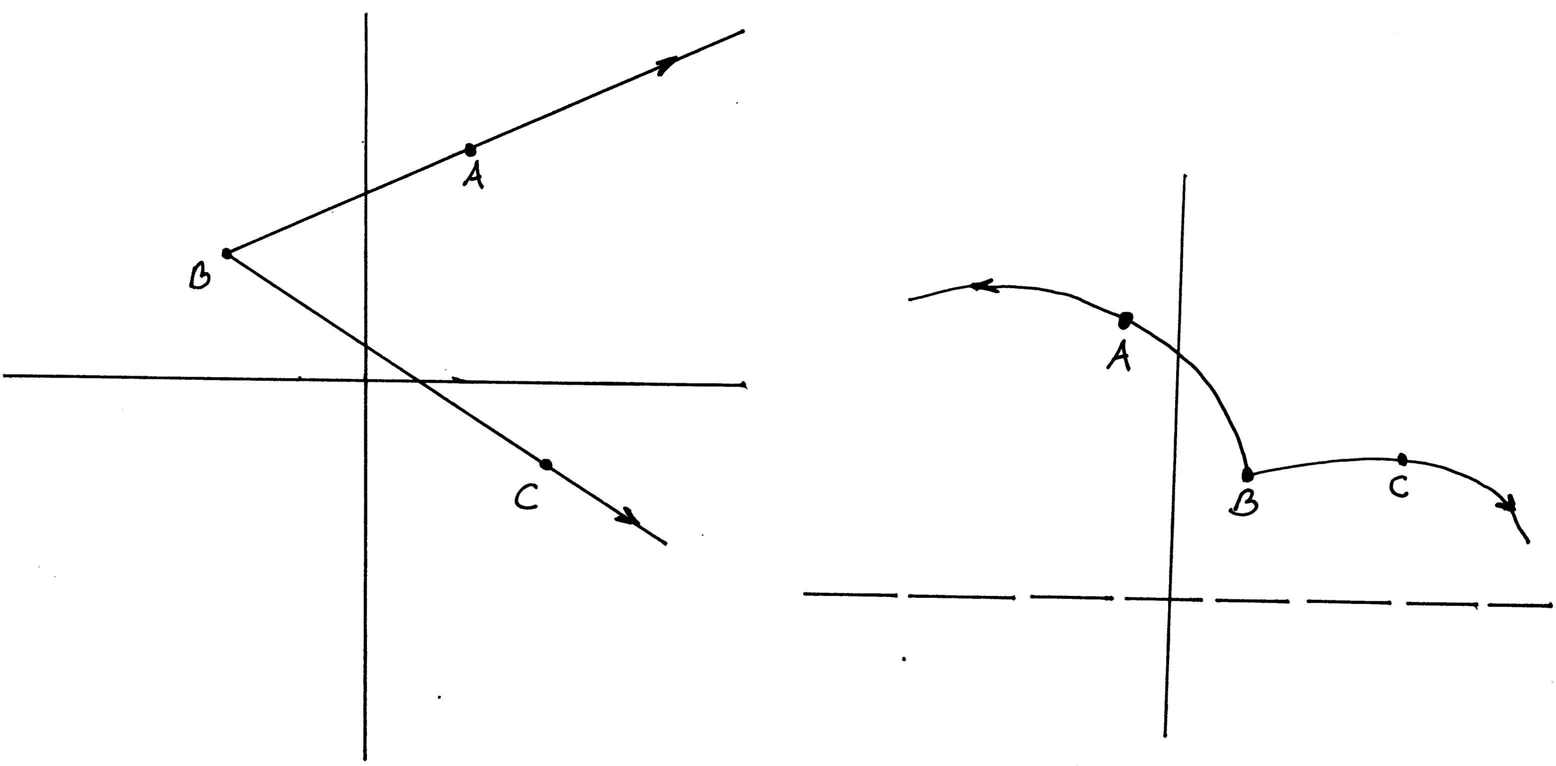
Uglovi i trouglovi

Važno je da primjetimo da ćemo ugao posmatrati kao skup, ne kao neki broj npr. 45°. Kasnije ćemo uvesti brojeve kao osobine uglova kad definišemo šta znači mjera ugla.

Definicija ($\measuredangle ABC$)

Ako su A, B i C tri nekolinearne tačke u metričnoj geometriji tada je ugao $\measuredangle ABC$ skup

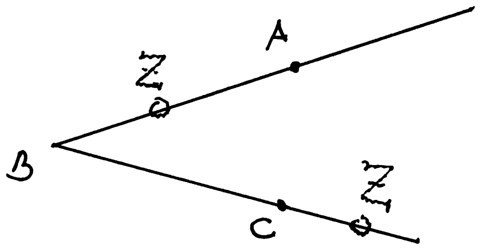
$$\measuredangle ABC = \overrightarrow{BA} \cup \overrightarrow{BC} = pp[B,A) \cup pp[B,C)$$



Primjeri uglova u Euklidovoj i Poincaré-ovoj ravni.

Lema

U metričnoj geometriji, B je jedina ekstremna tačka ugla $\measuredangle ABC$.



Skica dokaza:

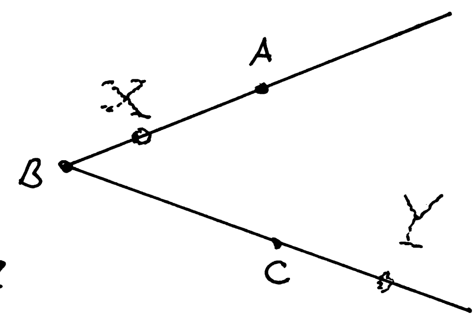
$Z \in \measuredangle ABC$ i $Z \neq B \Rightarrow Z$ prolazna tačka $\measuredangle ABC$

$Z \in \measuredangle ABC$ i $Z \neq B \Rightarrow$ ili $Z \in \overrightarrow{BA}$ ili $Z \in \overrightarrow{BC}$

$Z \in \overrightarrow{BA},\ Z \neq B \overset{\text{Teor.}}{\Longrightarrow} \overrightarrow{BA} = \overrightarrow{BZ} \Rightarrow \exists D \in \overrightarrow{BZ}\quad B-Z-D$

$D \in \overrightarrow{BA}$ i Z je između dvije tačke ugla $\measuredangle ABC$, B i D

B nije prolazna tačka $\measuredangle ABC$



$X-B-Y,\ X, Y \in \measuredangle ABC$

ili $X \in \overrightarrow{BA}$ ili $X \in \overrightarrow{BC}$

$X \in \overrightarrow{BA},\ X \neq B \overset{\text{Teor.}}{\Longrightarrow} \overrightarrow{BA} = \overrightarrow{BX}$

$Y-B-X \Rightarrow Y \notin \overrightarrow{BX} = \overrightarrow{BA} \quad \ldots (*)$

$Y \in \measuredangle ABC \overset{(*)}{\Longrightarrow} Y \in \overrightarrow{BC} \Rightarrow \overrightarrow{BC} = \overrightarrow{BY}$

$$\left.\begin{aligned} &A \in \overrightarrow{BA} = \overrightarrow{BX} \subseteq \overleftrightarrow{XY} \\ &B \in XY \\ &C \in \overrightarrow{BC} = \overrightarrow{BY} \subseteq \overleftrightarrow{XY} \end{aligned}\right\} \Rightarrow A, B \text{ i } C \text{ kolinearne}$$

#kontradikcija

Teorema ($\sphericalangle ABC = \sphericalangle DEF \Rightarrow B = E$)

U metričkoj geometriji, ako je $\sphericalangle ABC = \sphericalangle DEF$ tada je $B = E$.

dokaz:

$$\{B\} = \{Z \in \sphericalangle ABC \mid Z \text{ je ekstremna tačka ugla } \sphericalangle ABC\}$$
$$= \{Z \in \sphericalangle DEF \mid Z \text{ je ekstremna tačka ugla } \sphericalangle DEF\}$$
$$= \{E\}.$$

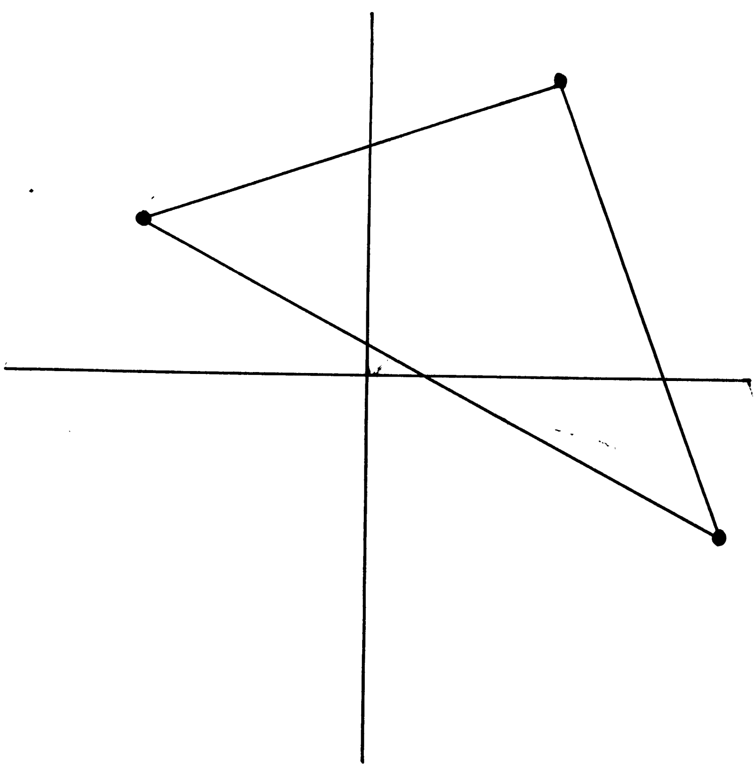
Definicija (vrh ugla $\sphericalangle ABC$)

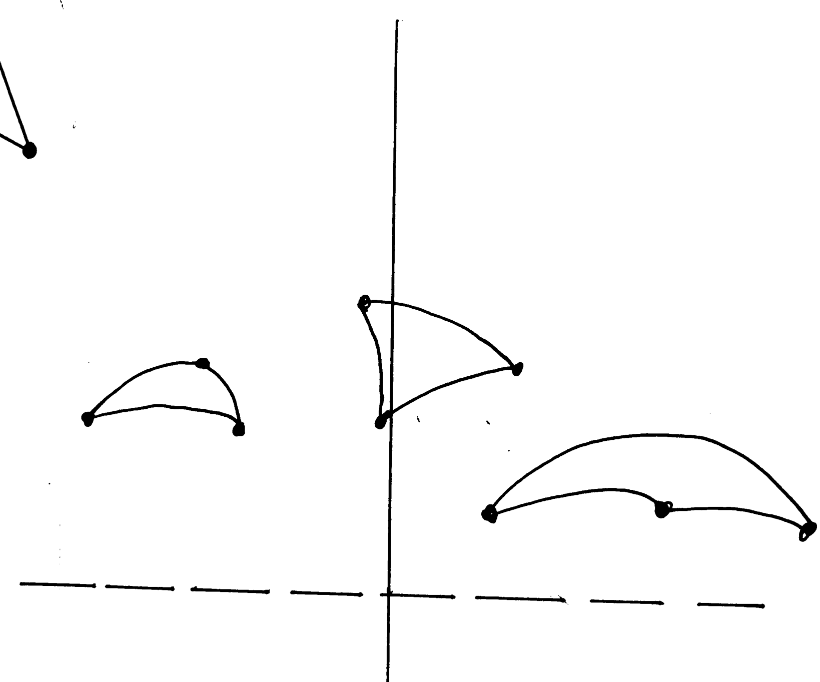
Vrh ugla $\sphericalangle ABC$ u metričnoj geometriji je tačka B.

Definicija (ΔABC)

Ako su $\{A, B, C\}$ nekolinearne tačke metrične geometrije tada je trougao ΔABC skup

$$\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$$

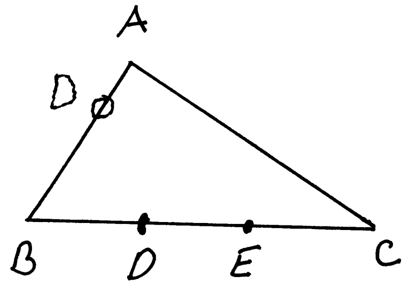




Primjeri trouglova u Euklidovoj i Poincaré-ovoj ravni.

Lema

U metričnoj geometriji, ako su A, B i C tri nekolinearne tačke tada je A ekstremna tačka trougla $\triangle ABC$.



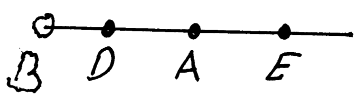
Skica dokaza:

$D-A-E,\ D,E \in \triangle ABC \Rightarrow D, E \in \overline{BC}$

$D \in \overline{AB} \Rightarrow$ ili $D = B$ ili $D \neq B$

$\Downarrow$ $E-A-B$ $\qquad$ $\Downarrow$ $E-A-D-B$ i $E-A-B$

$(D \neq A,\ D-A-E)$

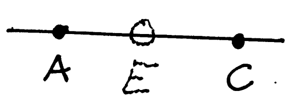


$\Rightarrow E \notin \overline{AB}$

$E \in \overline{AC}$ ili $E \in \overline{BC} \Rightarrow$ ili $C-E-A-B$ ili $C = E$

$\Downarrow$

$C-A-B$



$\Rightarrow A, B, C$ kolinearne

#kontradikcija (A, B i C su nekolinearne)

$\Rightarrow D \notin \overline{AB}$ ($\Rightarrow E \in \overline{AB}, \overline{AC}$ ili $\overline{BC}$)

Slično $D \notin AC$.

$D \in \triangle ABC \Rightarrow D \in \overline{BC} \overset{\text{slično}}{\Rightarrow} E \in \overline{BC} \Rightarrow D, E \in \overleftrightarrow{BC}$

$D-A-E \Rightarrow A \in \overleftrightarrow{BC} \Rightarrow A, B, C$ kolinearne

#kontradikcija

Tačka A ne može biti tačka između dvije tačke trougla $\triangle ABC$.

(#) Pokazati da ako je $\Delta ABC = \Delta DEF$ u metričkoj geometriji tada $\overleftrightarrow{AB} = p(A,B)$ sadrži tačno dvije od tački D, E i F.

Rj:

$$\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA} =$$

$$= \{M \in \mathcal{S} \mid A-M-B \text{ ili } B-M-C \text{ ili } C-M-A \text{ ili } M=A \text{ ili } M=B \text{ ili } M=C\}$$

$$\Delta DEF = \overline{DE} \cup \overline{EF} \cup \overline{FD} =$$

$$= \{N \in \mathcal{S} \mid D-N-E \text{ ili } E-N-F \text{ ili } F-N-D \text{ ili } N=D \text{ ili } N=E \text{ ili } N=F\}$$

$$\overleftrightarrow{AB} = \{R \in \mathcal{S} \mid R-A-B \text{ ili } A-R-B \text{ ili } A-B-R \text{ ili } R=A \text{ ili } R=B\}$$

$$\overleftrightarrow{AB} \cap \overline{AB} = \overline{AB} \quad \xRightarrow{\overline{AB} \subseteq \Delta ABC} \quad \overline{AB} \cap \overleftrightarrow{AB} \subseteq \Delta ABC \cap \overleftrightarrow{AB}$$

$$\overline{AB} \subseteq \overleftrightarrow{AB} \cap \Delta ABC$$

$$\Downarrow \ \Delta ABC = \Delta DEF$$

$$\overline{AB} \subseteq \Delta DEF$$

Posmatrajmo sad presjek $\overleftrightarrow{AB} \cap \{D,E,F\}$.

1° $\overleftrightarrow{AB} \cap \{D,E,F\} = \emptyset$

Tada $\overline{AB} \cap \{D,E,F\} = \emptyset \Rightarrow A,B \notin \{D,E,F\}$

i $M \notin \{D,E,F\} \ \forall M \in \{N \in \mathcal{S} \mid A-N-B\}$

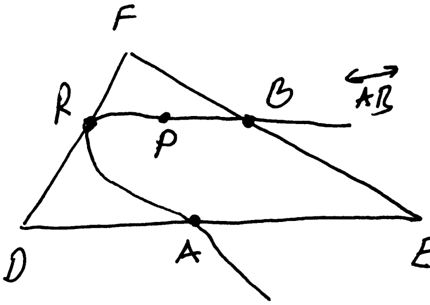
$$A,B \in \Delta ABC = \Delta DEF \quad \xRightarrow{\overline{AB} \cap \{D,E,F\} = \emptyset} \quad \begin{array}{l} A \in \{M \in \mathcal{S} \mid D-M-E \text{ ili } E-M-F \text{ ili } F-M-D\} \\ B \in \{M \in \mathcal{S} \mid D-M-E \text{ ili } E-M-F \text{ ili } F-M-D\} \end{array}$$

Moguć je jedan od sljedećih 6 slučajeva:

(i) $D-A-E$; $D-B-E$

Tada $D, E \in \overleftrightarrow{AB}$ #kontradikcija
($\overleftrightarrow{AB} \cap \{D,E,F\} = \emptyset$)

(ii) $D-A-E$; $E-B-F$



$\overline{AB} \subseteq \triangle DEF$ to tačku R za koju $A-R-B$ vrijedi da $R \in \triangle DEF$.

- $R \in \{D,E,F\} \Rightarrow \overleftrightarrow{AB} \cap \{D,E,F\} \neq \emptyset$ #kontradikcija
- $D-R-E \Rightarrow D, E \in \overleftrightarrow{AB}$ #kontradikc
- $E-R-F \Rightarrow E, F \in \overleftrightarrow{AB}$ #kontradikcija
- $D-R-F$

Izaberimo tačku P takvu da $R-P-B$

Kako $\overline{AB} \subseteq \triangle DEF \Rightarrow P \in \triangle DEF$.
Iz istog razloga kao iznad imamo da $D-P-F$

Ali sad
$$\left.\begin{array}{l} D-R-F \\ D-P-F \\ P, R \in \overleftrightarrow{AB} \end{array}\right\} \Rightarrow D, F \in \overleftrightarrow{AB}$$
#kontradikcija

Prema tome ne može vrijediti $D-A-E$ i $E-B-F$.

(iii) $D-A-E$; $F-B-D$

⋮ ZAVRŠITI ZA VJEŽBU